### Appendix A: Group theory of $D_4$

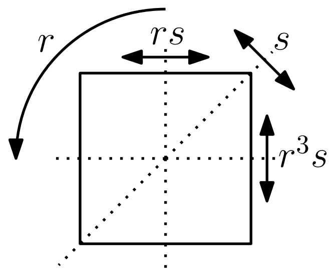



FIG. 1. $D_4$ realizes the symmetries of the square. As a semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, the vertical and horizontal reflections ($rs$ and $r^3s$) are exchanged under the diagonal reflection $s$.

The group $D_4$ can be defined abstractly as generated by elements $r$ and $s$ which satisfy $r^4 = s^2 = (sr)^2 = 1$. As symmetries of the square, $r$ rotates the square by $90°$ degrees and $s$ performs a diagonal reflection, as shown in Fig. 1. The group $D_4$ can also be seen as a semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the vertical and horizontal reflections $rs$ and $r^3s$ are swapped under diagonal symmetry $s$.

The group admits five irreducible representations (irreps). Other than the trivial irrep **1** and the faithful two-dimensional irrep **2** where

$$r = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \text{and} \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A1}$$

there are three sign representations, which we will label $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$. Each sign rep is uniquely defined by its "kernel", the subgroup on which the sign rep acts trivially.

1. $\boldsymbol{s}_1$ has kernel $\{1, r, r^2, r^3\}$ meaning it is represented by $s = rs = r^2 = r^3s = -1$,
2. $\boldsymbol{s}_2$ has kernel $\{1, r^2, s, r^2s\}$ meaning it is represented by $r = r^3 = rs = r^3s = -1$,
3. $\boldsymbol{s}_3$ has kernel $\{1, r^2, rs, r^3s\}$ meaning it is represented by $r = r^3 = s = r^2s = -1$.

The group also admits five conjugacy classes, $[1] = \{1\}$, $[r^2] = \{r^2\}$, $[r] = \{r, \bar{r}\}$, $[s] = \{s, \bar{r}^2s\}$ $[rs] = \{rs, r^3s\}$.

### Appendix B: Correspondence between anyons of bilayer Toric Code and anyons of $D_4$ TO

Mathematically, the anyons in the $G$ topological order (TO) corresponds to irreducible representations of the quantum double $\mathcal{D}(G)$. Each anyon can be given two labels: a conjugacy class $[g]$ and an irreducible representation of its centralizer $\pi_g$. A pure charge corresponds to the trivial conjugacy class with a choice of an irrep of $G$, while a pure flux corresponds a trivial irrep and a choice of conjugacy class. The quantum dimension of the anyon is given by the size of the conjugacy class times the dimension of the irrep. For $G = D_4$ enumerating all the possible choice of conjugacy classes and irreps gives a total of 22 anyons.

We are particularly interested in abelian anyons of $\mathcal{D}(D_4)$ and how they are related to the anyons that we measure in the toric code bilayer construction: $e_1e_2$, $m_1m_2$, and $s$. A complete treatment of how anyons map under gauging (which is implemented in this particular instance by measurement) can be found in Ref. 1.

First, without loss of generality we take the swap symmetry to be represented by the group element $s$. As shown in Fig. 1 it exchanges $rs$ and $r^3s$, the vertical and horizontal reflections. Since $rs$ and $r^3s$ generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, which is the kernel of $\boldsymbol{s}_3$, we identify the gauge charge of the swap symmetry with $\boldsymbol{s}_3$ as a gauge charge of the $D_4$ quantum double.

Next, we note that $e_1e_2$ is a gauge charge of the bilayer toric code. In particular, it should be an irrep of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In $D_4$, this is the subgroup generated by $rs$ and $r^3s$. Now, since $e_1e_2$ is the charged under the gauge transformation of both symmetries, while neutral under the diagonal symmetry $r^2$, it must therefore correspond to a representation where $rs = r^3s = -1$ while $r^2 = 1$. Moreover, $e_1e_2$ is neutral under the swap symmetry, meaning $s = 1$. Comparing to the irreps of $D_4$, we therefore see that this matches the irrep $\boldsymbol{s}_2$. By a similar argument, we find that $e_1e_2s$ corresponds to the irrep $\boldsymbol{s}_1$.

| Bilayer TC with SWAP symmetry ($\mathbb{Z}_2^2 \rtimes \mathbb{Z}_2$) | | | $D_4$ Quantum Double | | | dim |
|---|---|---|---|---|---|---|
| Orbit of anyon under SWAP | Stabilizer | SWAP charge | Conj class | Centralizer | irrep | |
| $[1]$ | $\mathbb{Z}_2$ | 1 | $[1]$ | $D_4$ | $\mathbf{1}$ | 1 |
| $[e_1e_2]$ | $\mathbb{Z}_2$ | $s$ | $[1]$ | $D_4$ | $\boldsymbol{s}_1$ | 1 |
| $[e_1e_2]$ | $\mathbb{Z}_2$ | 1 | $[1]$ | $D_4$ | $\boldsymbol{s}_2$ | 1 |
| $[1]$ | $\mathbb{Z}_2$ | $s$ | $[1]$ | $D_4$ | $\boldsymbol{s}_3$ | 1 |
| $[m_1m_2]$ | $\mathbb{Z}_2$ | 1 | $[r^2]$ | $D_4$ | $\mathbf{1}$ | 1 |
| $[f_1f_2]$ | $\mathbb{Z}_2$ | $s$ | $[r^2]$ | $D_4$ | $\boldsymbol{s}_1$ | 1 |
| $[f_1f_2]$ | $\mathbb{Z}_2$ | 1 | $[r^2]$ | $D_4$ | $\boldsymbol{s}_2$ | 1 |
| $[m_1m_2]$ | $\mathbb{Z}_2$ | $s$ | $[r^2]$ | $D_4$ | $\boldsymbol{s}_3$ | 1 |
| $[e_1] = \{e_1, e_2\}$ | $\mathbb{Z}_1$ | 1 | $[1]$ | $D_4$ | $\mathbf{2}$ | 2 |
| $[m_1] = \{m_1, m_2\}$ | $\mathbb{Z}_1$ | 1 | $[rs] = \{rs, r^3s\}$ | $\mathbb{Z}_2^2$ | $\mathbf{1}$ | 2 |

TABLE I. Correspondence between anyons of Bilayer TC along with the SWAP symmetry charge, and anyons of $D_4$ TO. (Certain anyons are omitted for simplicity.)

Finally, since $m_1m_2$ is a gauge flux of the bilayer toric code, it corresponds to a conjugacy class of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $m_1$ and $m_2$ are associated to group elements $rs$ and $r^3s$, their product is therefore $r^2$. Hence, $m_1m_2$ corresponds to the $r^2$ conjugacy class of $D_4$.

To conclude, the anyons we measure, $e_1e_2, m_1m_2, s$, generate eight anyons: $\{1, e_1e_2, m_1m_2, f_1f_2, s, e_1e_2s, m_1m_2s, f_1f_2s\}$. It is apparent that these anyons are all bosons and have trivial mutual braiding. Therefore, after gauging they are identified with eight abelian anyons of $D_4$ that forms a $\mathbb{Z}_2^3$ Lagrangian subgroup. The exact correspondence is summarized in Table I.

It is also worth pointing out how non-Abelian anyons are generated in this correspondence. First consider the anyon $e_1$, which corresponds to the irrep $(-1, 1)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Under the swap symmetry it transforms into $e_2$, corresponding to the irrep $(1, -1)$. Therefore, after gauging the swap symmetry, these two anyons combine into a single non-Abelian anyon with quantum dimension 2. This corresponds to the irrep $\mathbf{2}$ of $D_4$. Note that in this case, the non-trivial action on the anyons means that it is not meaningful to attach the charge of the swap symmetry onto $[e_1]$. Moreover, this can be interpreted as the fusion rule $\mathbf{2} \times \boldsymbol{s}_3 = \mathbf{2}$ for $D_4$ anyons. Similarly, the anyon $m_1$ and $m_2$ corresponds to the conjugacy class $\{rs\}$ and $\{r^3s\}$ respectively. After gauging, the conjugation of $s$ combines them into a single conjugacy class $[rs]$ of $D_4$, resulting in a non-Abelian gauge flux.

For further details on this specific correspondence, we refer to a thorough review in Sec. II of Ref. 2.

### Appendix C: Preparation of $D_4$ Topological order

Here we prove that the protocol in the main text indeed prepares the $D_4$ quantum double with a single round of measurement. We first define the protocol on the vertices edges and faces of the triangular lattice, where the preparation is most natural.

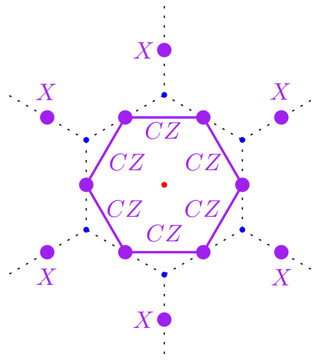
We place qubits on the vertices, edges and plaquettes of the triangular lattice as in Fig. 3a of the main text. For convenience, the protocol is reproduced here:

$$|D_4\rangle_E = \langle x|_{PV} \prod_{\langle v,e\rangle} CZ_{ve} \prod_v e^{\pm\frac{\pi i}{8} Z_v} H_v \prod_{\langle p,v\rangle} CZ_{pv} \,|+\rangle_{PEV} \,. \tag{C1}$$

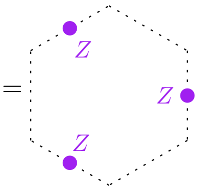
Namely, we start with a product state $|+\rangle$ for all qubits, apply the above quantum circuit, and perform projective measurements in the $x$-basis, where $x = \pm$ labels the measurement outcomes on each vertex and plaquette. Here, the $\pm$ sign in $e^{\pm\frac{\pi i}{8} Z_v}$ denotes that the phase gate we perform takes an alternating sign depending on the vertex sublattice (colored red or orange in in Fig. 3a of the main text).

The final state prepared, after a further Hadamard on all edges, is conveniently described as the simultaneous +1 eigenstate of the following "stabilizers"[3] defined for each vertex

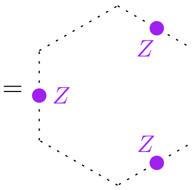
$$\boldsymbol{A}_p = x_p \times \prod_{v\subset p} x_v \times$$




$$, \quad \boldsymbol{B}_p^{(1)} =$$




$$, \quad \boldsymbol{B}_p^{(2)} =$$




$$. \tag{C2}$$

as we will momentarily derive. Note that without the $CZ$ operators in $\boldsymbol{A}_p$, these describe the stabilizers of three copies of the toric code. The $CZ$ operators couple the toric code in a non-trivial way that creates the $D_4$ TO (see Appendix E 1 for a further relation to the $\mathbb{Z}_2^3$ twisted quantum double and SPT phases).

In this model, although $\boldsymbol{B}_p^{(i)}$ commutes with all operators, two adjacent $\boldsymbol{A}_p$ operators only commute up to some product of $\boldsymbol{B}_{p'}^{(i)}$. For example, consider two adjacent plaquettes $p_L$ and $p_R$ sharing a vertical edge, then one has

$$\boldsymbol{A}_{p_L}\boldsymbol{A}_{p_R} = \boldsymbol{B}_{p_L}^{(1)}\boldsymbol{B}_{p_R}^{(2)}\boldsymbol{A}_{p_R}\boldsymbol{A}_{p_L}. \tag{C3}$$

Nevertheless, one can still have a unique state which has eigenvalue $+1$ under all the above operators simultaneously, which is the state we prepare.

To facilitate in showing the above claim, we split the process into three steps

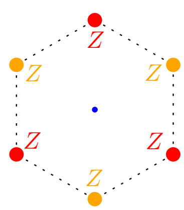
$$|D_4\rangle_E = \langle x|_V \prod_{\langle p,e\rangle} CZ_{pe}\,|+\rangle_E \times e^{\pm\frac{\pi i}{8}Z_v}\prod_v H_v \times \langle x|_P \prod_{\langle v,p\rangle} CZ_{vp}\,|+\rangle_{PV} \tag{C4}$$

The first step involves create a cluster state on the dice lattice by connecting each plaquette to each of the six vertices. Measuring all the plaquettes in the $x$-basis creates the color code. That is, the state

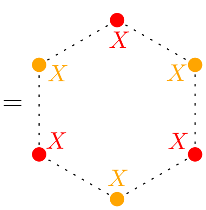
$$|\mathrm{CC}\rangle = \langle x|_P \prod_{\langle v,p\rangle} CZ_{vp}\,|+\rangle_{PV} \tag{C5}$$

is the $+1$ eigenstate of the stabilizers

$$A_p = x_p \times$$




$$, \qquad B_p =$$




$$\tag{C6}$$

For convenience, let us denote the six vertices surrounding the plaquettes as $1,\ldots,6$. Then,

$$A_p = x_p \times \prod_{n=1}^{6} Z_n, \qquad B_p = \prod_{n=1}^{6} X_n \qquad A_pB_p = -x_p \times \prod_{n=1}^{6} Y_n \tag{C7}$$

Here, we have also included the product stabilizer to point out that the state inherently has a $\mathbb{Z}_2$ symmetry that swaps $A_p$ and $A_pB_p$ *independent* of the measurement outcome. This symmetry is realized by acting with $\frac{Y+Z}{\sqrt{2}}$ on one sublattice and $\frac{Y-Z}{\sqrt{2}}$ on the other sublattice. That is, on the six sites, it acts as $\prod_{n=1}^{6}\frac{Y+(-1)^nZ}{\sqrt{2}}$. (this sublattice structure is essential to obtain the minus sign in $A_pB_p$).

In order to measure the Gauss law for this symmetry, it is helpful to perform a basis transformation to turn the symmetry $\prod_v \frac{Y\pm Z}{\sqrt{2}}$ (where $\pm$ denotes the sublattice structure) into $\prod_v X_v$. This is accomplished by the second layer of the protocol: $\prod_v e^{\pm\frac{\pi i}{8}Z_v}H_v$ After the transformation, the state is given by stabilizers

$$\tilde{A}_p = x_p \times \prod_{n=1}^{6}\frac{X_n+(-1)^nY_n}{\sqrt{2}} \qquad \tilde{B}_p = \prod_{n=1}^{6} Z_n, \qquad \tilde{A}_p\tilde{B}_p = x_p \times \prod_{n=1}^{6}\frac{X_n-(-1)^nY_n}{\sqrt{2}}. \tag{C8}$$

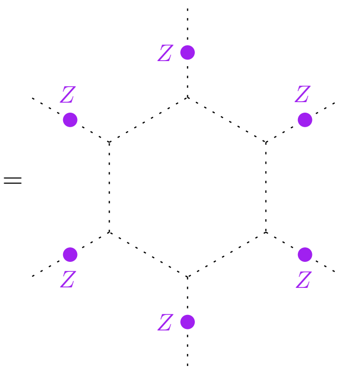
Indeed, $\prod_v X_v$ swaps $\tilde{A}_p$ and $\tilde{A}_p\tilde{B}_p$ while leaving $\tilde{B}_p$ invariant as desired.

Finally, in the last step we measure the Gauss law for this $\mathbb{Z}_2$ symmetry $X_v \prod_{e\supset v} Z_e$ on all vertices.

This can be done by initializing qubits on all edges in the $|+\rangle$ state, applying Controlled-$Z$ connecting vertices to all the nearest edges and measuring all the vertices in the X basis.

The new edges introduced are stabilized by $X_e$, and after applying $\prod_{v,e} CZ_{ve}$, the stabilizers are given by $\tilde{A}_p C_p$ and $\tilde{B}_p$ for each plaquette where

$C_p =$




(C9)

and $D_e = Z_v X_e Z_{v'}$ for each edge, where $v$ and $v'$ are the vertices at the end points of $e$. Now, to perform the measurement in the $X$ basis on all vertices, we need to find combinations of stabilizers that commute with the measurement. First, we note the following combinations do not involve vertex terms

$$B_p D_{12} D_{34} D_{56} = X_{12} X_{34} X_{56} \qquad\qquad B_p D_{23} D_{45} D_{61} = X_{23} X_{45} X_{61}. \tag{C10}$$

and therefore survives the measurement. Next, we consider the symmetric combination

$$\tilde{A}_p C_p \frac{1+\tilde{B}_p}{2} = x_p \times \prod_{n=1}^{6} X_n \times C_p \times \left[ \prod_{n=1}^{6} \frac{1+(-1)^n i Z_n}{\sqrt{2}} + \prod_{n=1}^{6} \frac{1-(-1)^n i Z_n}{\sqrt{2}} \right] \tag{C11}$$

Expanding the bracket we find

$$\tilde{A}_p C_p \frac{1+\tilde{B}_p}{2} = x_p \times \prod_{n=1}^{6} X_n \times C_p \times \frac{1+\tilde{B}_p}{2} \prod_{n=1}^{6} \frac{1+Z_{n-1}Z_n + Z_n Z_{n+1} - Z_{n-1}Z_{n+1}}{2} \tag{C12}$$

Since the state satisfies $\tilde{B}_p = 1$, it therefore also has eigenvalue $+1$ under the "stabilizer"

$$x_p \times \prod_{n=1}^{6} X_n \times C_p \times \prod_{n=1}^{6} \frac{1+Z_{n-1}Z_n + Z_n Z_{n+1} - Z_{n-1}Z_{n+1}}{2} \tag{C13}$$

Note that these "stabilizers" no longer commute amongst themselves. Now, using the fact that the state satisfies $Z_n X_{n,n+1} Z_{n+1} = 1$, we can replace $Z_n Z_{n+1}$ by$X_{n,n+1}$. This results in

$$x_p \times \prod_{n=1}^{6} X_n \times C_p \times \prod_{n=1}^{6} \frac{1+X_{n-1,n} + X_{n,n+1} - X_{n-1,n}X_{n,n+1}}{2} \tag{C14}$$

This "stabilizer" now commutes with the measurement on all vertices. With measurement outcomes $X_n = x_n = \pm 1$. To conclude, the final "stabilizers" for each plaquette are

$$x_p \times \prod_{n=1}^{6} x_n \times C_p \times \prod_{n=1}^{6} \frac{1+X_{n-1,n} + X_{n,n+1} - X_{n-1,n}X_{n,n+1}}{2}, \qquad X_{12}X_{34}X_{56}, \qquad X_{23}X_{45}X_{61}. \tag{C15}$$

Finally, performing Hadamard on all edges and using the fact that $CZ_{ij} = \frac{1+Z_i+Z_j-Z_iZ_j}{2}$, we recover the "stabilizers" in Eq. (C2).

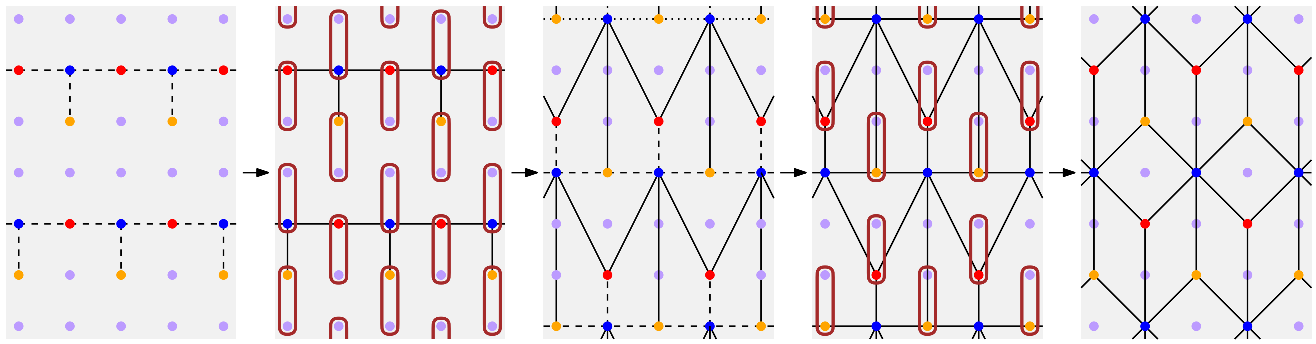


FIG. 2. Preparation of the dice lattice cluster state in Sycamore using SWAP gates. Purple qubits only participate in the swapping procedure, and are not part of the cluster state.

## Appendix D: Implementation on Sycamore

First, let us count the depth of the 2-body gates required on the ideal lattice in Fig. 3a of the main text. The dice lattice cluster state can be prepared in depth 6, while the heavy-hex lattice cluster state can be prepared in depth 3. This gives a total 2-body depth count of 9.

Next, we discuss the details of implementation on the a quantum processor with connectivity of the square lattice, such as Google's Sycamore quantum chip. The first step in our protocol $\prod_{\langle p,v\rangle} CZ_{pv}$ requires preparing a cluster state on the dice lattice. This can be achieved by the help of SWAP gates. As seen in Fig. 2, the four steps corresponds to steps 1,2,3 and 5 in Fig. 3b, and indeed produces the cluster state.

Next, we note that the single site rotation $e^{\pm\frac{\pi i}{4}Z_v}H_v$ can be pulled back through the final layer of SWAP gates, so that it acts on the corresponding sites *before* the swap. This results in in step 4 of Fig. 3b. Lastly, $\prod_{\langle v,e\rangle} CZ_{ve}$ which forms the heavy-hex lattice is implemented in step 6.

To count the number of gates used, the $CZ$ gates in steps 1 and 3 can each be implemented in depth-3. Therefore, the 2-body gate depth count for the six steps combined is is 3+1+3+0+1+3=11.

More practically, we should count the 2-body gate depth using the innate gates of the Sycamore processor. In particular, the SWAP gate can be decomposed in to three $CZ$ gates interspersed by Hadamard gates. Conveniently, one of the $CZ$ gates from the SWAP in step 5 exactly cancels one of the $CZ$ layers in step 6. Thus, the innate 2-body depth count is 3+3+3+0+2+2=13.

## Appendix E: Single-round preparation of topological orders with Lagrangian subgroup

We give a formal argument that any non-Abelian topological order in two spatial dimensions which admits a Lagrangian subgroup can be prepared using a single round of measurements.

To recall, a Lagrangian subgroup $A$ is a subset of Abelian anyons that are closed under fusion, have trivial self and mutual statistics, and that every other anyon braids non-trivially with at least one of the anyons in the subgroup[4, 5]. Note that the full set of anyons describing the theory does not need to be Abelian[1]. Before moving forward, we remark that $A$ can serve two purposes in this discussion: it can be a set that contains anyons, or can also function as an abstract group.

Given a topological order and a Lagrangian subgroup $A$, one can "condense" [6] all the anyons in $A$. To do this, we introduce an auxiliary system with global symmetry given by the group $A$. The system has charges that transform under irreps of the global symmetry $a_{\mathrm{phys}}$. Note that these charges are physical, unlike the anyons $a$ which are gauge charges of an unphysical gauge group. Next, one performs a condensation for all bound states $a\times a_{\mathrm{phys}}^{-1}$. This identifies $a\sim a_{\mathrm{phys}}$ in the ground state of the condensed phase. The symmetry of the system is still $A$. However, the remaining anyons are confined, since they braid non-trivially with the anyons that are condensed. These confined anyons now serve as defects of the symmetry $A$. Since the resulting phase no longer has any anyons, it is therefore a (bosonic) Symmetry-Protected Topological (SPT) phase with global symmetry $A$. Let us call this state $|\psi_{\mathrm{SPT}}\rangle$ This process is also known as gauging the 1-form symmetry for all anyon lines in $A$[4, 7][2]

[1] In fact, we use the word *subgroup* in contrast to the more general *subalgebra* precisely because we restrict $A$ to only contain Abelian anyons.

Now, we provide a protocol to prepare such a topological order. We start from a trivial product state with symmetry group $A$. It is known that any SPT phase in two spatial dimensions can be prepared (by temporarily breaking the symmetry) with a finite-depth local unitary[9][3]. Therefore, after preparing $|\psi_{\mathrm{SPT}}\rangle$, we measure the symmetry charges of $A$ by coupling the charges to ancillas so that we can measure its Gauss law. Note that this is nothing but the protocol to implement the Kramers-Wannier transformation in Ref. 8. After the measurement, the charges of $|\psi_{\mathrm{SPT}}\rangle$ are promoted to gauge charges, and therefore realizes the anyons in $A$, and the symmetry fluxes are promoted to deconfined gauge fluxes, restoring the remaining anyons in the theory. In other words, our measurement has reversed the condensation by gauging the global symmetry $A$. To summarize, we have used finite-depth local unitaries and one round of measurement to prepare a state in the desired phase without feedforward or postselection. Note that if one moreover wants to prepare exactly the ground state of the phase, a single round of feedforward gates suffices to pair up all the anyons in $A$ that result from the measurement.

The condition of a Lagrangian subgroup can actually be relaxed if we allow physical fermions as resources. In particular, the subgroup $A^f$ can now contain anyons that have fermionic self-statistics, a fermionic Lagrangian algebra[10]. In this case, one performs "fermion condensation"[11]. For any fermionic anyon in $A^f$ the bound state that one condenses is now $f \times f_{\mathrm{phys}}^{-1}$. This gives a fermionic SPT state $|\psi^f_{\mathrm{SPT}}\rangle$ with global symmetry $A^f$, which contains fermion parity as a subgroup.

Similarly, starting with a trivial product state with fermionic symmetry $A^f$ it is possible to prepare the SRE state $|\psi^f_{\mathrm{SPT}}\rangle$ using finite-depth local unitaries. Then, one measures the Gauss law for this symmetry. For group elements that corresponds to anyons with fermionic statistics, measuring the Gauss law of the fermion amounts to performing the two-dimensional Jordan-Wigner transformation (bosonization)[12], which can be performed using measurements[8].

### 1. Example: $D_4$ TO revisited

To give a concrete example, let us consider the quantum double for $D_4$. The Lagrangian subgroup is given by the sign representations along with the conjugacy class of the center. These anyons form a group $A = \mathbb{Z}_2^3$. By performing condensation, one arrives at an invertible state with symmetry $\mathbb{Z}_2^3$. In fact, this state is a Symmetry-Protected Topological state, and can be deformed (while preserving the symmetry) to following hypergraph state [3]

$$|\psi^{D_4}_{\mathrm{SPT}}\rangle = \prod_{\langle p_1 p_2 p_3\rangle} CCZ_{p_1 p_2 p_3} \, |+\rangle_P \tag{E1}$$

where $\langle p_1 p_2 p_3\rangle$ denotes three plaquettes that share a common vertex[4]. To describe the $\mathbb{Z}_2^3$ symmetry, we first note the plaquettes are three-colorable (say, red green and blue), such that no two adjacent plaquettes have the same color. Then each $\mathbb{Z}_2$ symmetry acts as spin flips on a plaquette of a fixed color.

To prepare the $D_4$ TO, we thus first prepare the above hypergraph state using $CCZ$. Then, we gauge the $\mathbb{Z}_2^3$ symmetry by measuring the Gauss law $X_p \prod_{e\in} Z_e$, where $e$ are the six edges radiating out of each plaquette. This can be done via

$$|D_4\rangle_E = \langle +|_P \prod_{p,e\in} CZ_{pe} \, |+\rangle_E \, |\psi^{D_4}_{\mathrm{SPT}}\rangle \tag{E2}$$

It is shown in Ref. 3 that the resulting state (after applying Hadamard on all edges) has exactly the same "stabilizers" we derived in Eq. (C2). This corresponds to the fact that the $D_4$ TO can be regarded as a $\mathbb{Z}_2^3$ twisted quantum double[13–15].

### 2. Example: $Q_8$ TO

As a second example, we consider the quantum double for the quaternion group $Q_8$. The Lagrangian subgroup also consists of sign representations along with the conjugacy class of the center, and forms the group $A = \mathbb{Z}_2^3$. After

[2] We remark that if one wishes, this condensation process can be implemented physically without tuning through a phase transition using measurements [8]. The auxiliary degrees of freedom serve as ancillas for which the hopping that promotes the condensation can be measured.

[3] up to possibly the $E_8$ phase, which ultimately decouples from the desired topological order

[4] This SPT corresponds to the so-called type-III cocycle $a_1 a_2 a_3$.

condensation, we arrive at a different SPT state, corresponding to the following hypergraph state

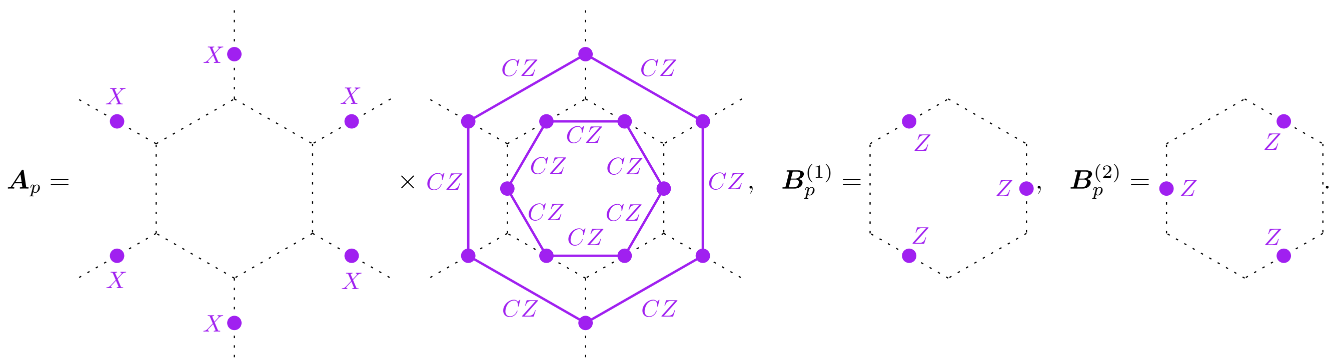
$$|\psi^{Q_8}_{\mathrm{SPT}}\rangle = \prod_{\langle p^C_1 p^C_2 p^C_3\rangle} CCZ_{p^C_1 p^C_2 p^C_3} \prod_{\langle p_1 p_2 p_3\rangle} CCZ_{p_1 p_2 p_3} \, |+\rangle_P \tag{E3}$$

where $\langle p^C_1 p^C_2 p^C_3\rangle$ consists of three plaquettes of the same color connected by edges in a triangle shape[5].

Thus, the $Q_8$ TO can be prepared as

$$|Q_8\rangle_E = \langle +|_P \prod_{p,e\in} CZ_{pe} \, |+\rangle_E \, |\psi^{Q_8}_{\mathrm{SPT}}\rangle \tag{E4}$$

and the "stabilizers" of this state after Hadamard is given by




(E5)

### 3. Example: Double Ising TO

As a final example, we discuss how to prepare the Doubled Ising TO, which is obtained by stacking the Ising TO consisting of anyons $\{1, \sigma, \epsilon\}$ with its time-reversed partner $\{1, \bar{\sigma}, \bar{\epsilon}\}$. Since $\epsilon$ and $\bar{\epsilon}$ are fermions, a Lagrangian subgroup does not exist. Nevertheless, it does have a fermionic Lagrangian subgroup $A^f = \{1, \epsilon, \bar{\epsilon}, \epsilon\bar{\epsilon}\}$. Condensing the fermionic Lagrangian subgroup results in an SPT state with $\mathbb{Z}_2 \times \mathbb{Z}^F_2$ symmetry. The precise wavefunction of this SPT state can be found in Ref. 16, and since it is short-range entangled, it can be prepared with a finite-depth quantum circuit.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[5] This SPT can be described by a combination of type-I and type-III cocycles $a_1^3 + a_2^3 + a_3^3 + a_1 a_2 a_3$ [14].